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E&M - Exam 6

Question 1

5/11

$n_1 = 1.3$
$n_2 = 1$

(1.) $R_n = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{0.3}{2.3}\right)^2 \doteq \underline{0.017}$

(2.) Yes, there is (*) but only for p-polarization

p-polar.: $R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 \Rightarrow R = 0 \Leftrightarrow \alpha = \beta$

$$\frac{\cos\theta_T}{\cos\theta_I} = \frac{n_2}{n_1}$$

(*) called Brewster's angle.

$$\Rightarrow \tan\theta_B = \frac{n_2}{n_1} = \frac{1.3}{1}$$

$$\theta_B \doteq 52.43^\circ$$

$R = 0$ for s-polarization when $\theta_I = \theta_T$ but this would mean $\sin\theta_T = \sin\theta_I$ hence also $n_1 = n_2$ - which is not the case

For s-polarization, there is no such angle ⇒ for explanation see next page!

$R = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2 \Rightarrow R = 0 \Leftrightarrow 1 = \alpha\beta \Leftrightarrow \alpha = \frac{1}{\beta} \Leftrightarrow \frac{\cos\theta_T}{\cos\theta_I} = \frac{n_1}{n_2}$

$\frac{n_1}{n_2} = \frac{\sin\theta_T}{\sin\theta_I}$

$\frac{\sin\theta_T}{\sin\theta_I} = \frac{\cos\theta_T}{\cos\theta_I} \Leftarrow \frac{\sin\theta_I}{\cos\theta_I} = \frac{\sin\theta_T}{\cos\theta_T} \Leftrightarrow \tan\theta_I = \tan\theta_T$

hence $\theta_I = \theta_T$

(3.) reflection becomes 100% at the angle of 90° for both polarisations:

(p-) $R = 1 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2$; $\alpha \to \infty$ at 90° ⇒ $R = 1$ at $\theta_I = 90^\circ$

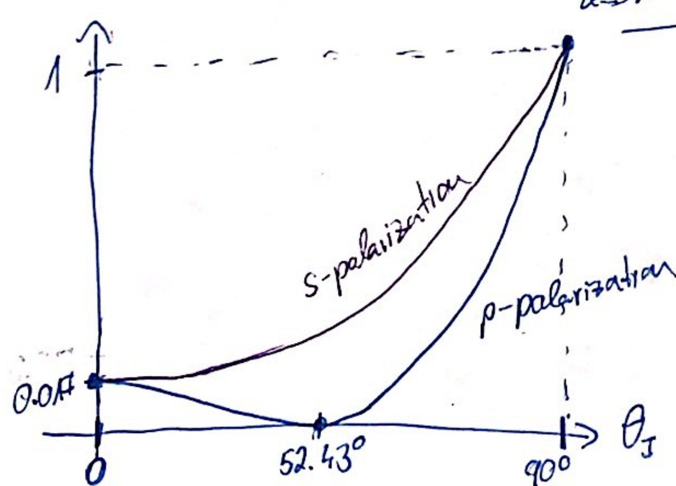
at 90°: $\alpha = \frac{\cos\theta_T}{\cos 90^\circ} = \frac{\cos\theta_T}{0} \to \infty$

$$\lim_{\alpha \to \infty} \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 = 1$$

(s-) $R = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2$ ⇒ $R = 1$ at $\theta_I = 90^\circ$ for both polarisations

$$\lim_{\alpha \to \infty} \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2 = 1$$

(4.)



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Question 2 15/15

$$\vec{K}(t) = \begin{cases} 0, & t \le 0 \\ K_0 \hat{z}, & t > 0 \end{cases}$$

retardation:

$r = \sqrt{s^2 + x^2}$

$t_r = \frac{r}{c} \geq 0 \quad t \geq \frac{r}{c}$ ⟹ $r \le ct$, $\sqrt{s^2+x^2} \le ct$, $s \le \sqrt{(ct)^2 - x^2}$ ⟹ $s_m = \sqrt{(ct)^2 - x^2}$

1. Show: $\vec{A}(x,t) = \frac{\mu_0 K_0 (ct - x)}{2} \hat{z}$, $x > 0$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}, t_r)}{r} da = \frac{\mu_0}{4\pi} \int \frac{K_0 \hat{z}}{\sqrt{s^2+x^2}} da = \frac{\mu_0 \hat{z}}{4\pi} \int_0^{2\pi}\int_0^{s_m} \frac{K_0}{\sqrt{s^2+x^2}} s\, ds\, d\phi =$$

$$= \frac{\mu_0 K_0 \hat{z}}{4\pi} \cdot 2\pi \int_0^{\sqrt{(ct)^2 - x^2}} \frac{1}{2} \frac{2s}{\sqrt{s^2+x^2}} ds = \frac{\mu_0 K_0 \hat{z}}{4} \cdot \frac{\sqrt{s^2+x^2}}{\frac{1}{2}} \Bigg|_{s=0}^{\sqrt{(ct)^2-x^2}} =$$

$$= \frac{\mu_0 K_0 \hat{z}}{2} \left(\sqrt{(ct)^2 - x^2 + x^2} - \sqrt{x^2} \right) = \frac{\mu_0 K_0 \hat{z}}{2} (ct - |x|) =$$

$$= \frac{\mu_0 K_0 (ct - x)}{2} \hat{z}, \text{ for } 0 < x < ct$$

2. $\vec{E}(x,t) = -\nabla V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 K_0 c}{2} \hat{z}$ for $0 < x < ct$, 0 otherwise (the "news" about current haven't reached the point yet)

$V = 0$ (no charges)

$$\vec{E} = \begin{cases} -\frac{\mu_0 K_0 c}{2} \hat{z}, & 0 < x < ct \\ 0, & x \geq ct \end{cases}$$

Also can be written as: $\vec{E} = -\frac{\mu_0 K_0 c}{2} H(ct - x) \hat{z}$ using Heaviside step function

3. $\vec{B}(x,t) = \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & A \end{vmatrix} = -\frac{\partial A(x)}{\partial x} \hat{y} = \frac{\mu_0 K_0}{2} \hat{y}$ for $0 < x < ct$

$$\Rightarrow \vec{B} = \begin{cases} \frac{\mu_0 K_0}{2} \hat{y}, & 0 < x < ct \\ 0, & x \geq ct \end{cases}$$

or equivallently: $\vec{B} = \frac{\mu_0 K_0}{2} H(ct - x) \hat{y}$

4.
- ✓ (the news) travel at c
- ✓ $(\vec{E} + \vec{B}) \perp \hat{x}$ – transverse and perpendicular to each other
- ✓ functions of $(ct - x)$ – although inside the step function
- ✓ $E_0 = cB_0$

↳ and they ✓ do satisfy the wave equation but only trivially and are not differentiable at $x = ct$. They qualify as elmag waves but aren't harmonic.

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Question 3 13/13

$\lambda = 500\,nm$

$h = 1\,mm$

$d \sim 1\,nm$

(1.) approximations made: $d \ll r$, $d \ll \lambda$, $r \gg \lambda$

↳ in this case $r \geq h$

i, $d \ll r$: $d \sim 10^{-9}\,m$
$h = 10^{-3}\,m \Rightarrow r \geq 10^{-3}\,m$

⇒ there is 6 ~~magnitud~~ orders of magnitude difference

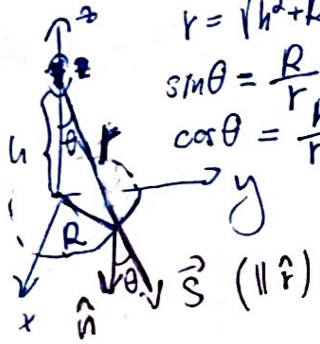
⇒ $d \ll r$ ✓

ii, $d \ll \lambda$: $d \sim 1\,nm$, $\lambda = 500\,nm$ ⇒ λ is ~500× larger than d.

This is enough to say that $d \ll \lambda$. ✓

iii, $r \gg \lambda$: $\lambda = 5 \cdot 10^{-7}\,m$
$h = 10^{-3}\,m \Rightarrow r \geq 10^{-3}\,m$

⇒ r is more than 2000× larger than λ, so it holds that $r \gg h$. ✓



$r = \sqrt{h^2 + R^2}$

$\sin\theta = \frac{R}{r}$

$\cos\theta = \frac{h}{r}$

(2.) Show: $I(R) = \frac{3\langle P\rangle}{8\pi} \frac{R^2 h}{(R^2+h^2)^{5/2}}$

$\langle P\rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$

$\langle \vec{S}\rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$

⇒ to get Intensity hitting the detector, we dot with normal vector (of the detector)

$$I(R) = \langle \vec{S}\rangle \cdot \hat{n} = \langle |\vec{S}|\rangle \cdot 1 \cdot \cos\theta =$$

$$= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \frac{\sin^2\theta}{r^2} \cdot \cos\theta = \frac{\langle P\rangle}{\frac{32}{12}\pi} \frac{R^2}{r^2 r^2} \frac{h}{r} =$$

$$= \frac{3\langle P\rangle}{8\pi} \frac{R^2 h}{(h^2+R^2)^{5/2}}$$ ■

(3.) $R_m = ?$

$$\frac{dI}{dR} = \frac{3\langle P\rangle h}{8\pi}\left(\frac{2R(h^2+R^2)^{5/2} - R^2 \cdot \frac{5}{2} \cdot 2R(h^2+R^2)^{3/2}}{(h^2+R^2)^5}\right) = 0$$ (to find extremes)

$$2R\sqrt{h^2+R^2}^{\,5} - 5R^3\sqrt{h^2+R^2}^{\,3} = 0 \quad \Big\} \; h^2+R^2 \neq 0$$

$$R\left(2\sqrt{h^2+R^2}^{\,2} - 5R^2\right) = 0$$

$$R(2h^2 + 2R^2 - 5R^2) = 0$$

$$R(2h^2 - 3R^2) = 0 \Rightarrow R = 0 \quad \vee \quad 3R^2 = 2h^2$$

$R = 0$ ⇓ along the axis of oscillations ⇒ minimum

$\Rightarrow R_m = \sqrt{\frac{2}{3}}\,h$ *

* $I \to 0$ for $R \to 0$ and $R \to \infty$ ⇒ ∃ max in between ⇒ only 1 extreme ∈ (0, ∞) found ⇒ maximum.

Intensity is maximal at $R_m = \sqrt{\frac{2}{3}}\,h$

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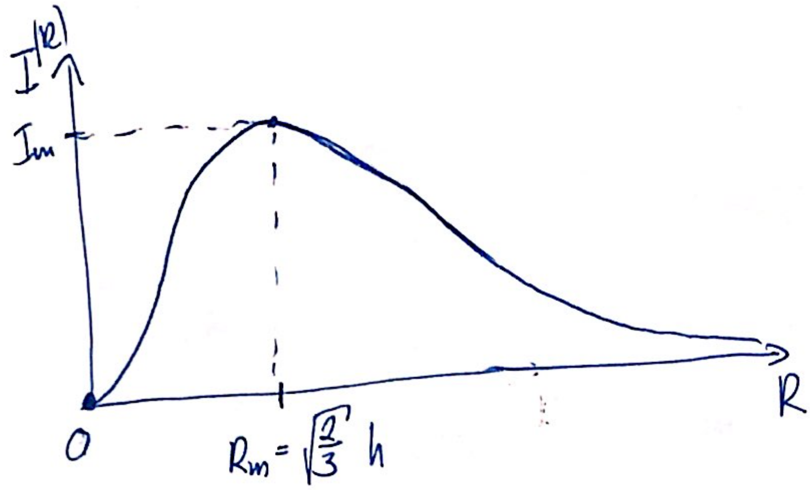
4.



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Question 1

2. s-polarization

$$R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 \Rightarrow R = 0 \Leftrightarrow 1 = \alpha\beta \Leftrightarrow \alpha = \frac{1}{\beta}$$

$$\alpha = \frac{1}{\beta} \Leftrightarrow \frac{\cos\theta_T}{\cos\theta_I} = \frac{n_1}{n_2} \quad \text{but also} \quad \frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

$$\Rightarrow \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sin\theta_T}{\sin\theta_I} \Leftrightarrow \frac{\sin\theta_T}{\cos\theta_T} = \frac{\sin\theta_I}{\cos\theta_I} \Leftrightarrow \tan\theta_T = \tan\theta_I \Leftrightarrow \theta_T = \theta_I$$

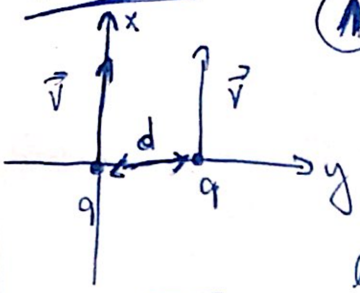
$R = 0$ at $\theta_T = \theta_I$ but this would imply $\frac{\sin\theta_T}{\sin\theta_I} = 1 = \frac{n_1}{n_2}$ but $n_1 \neq n_2$,

then $\sin\theta_T = \sin\theta_I \Rightarrow$

therefore we arrive at contradiction.

E&M 6 [redacted]

Question 4 11/12



$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

(1.) $\vec{E}_0$ ... in particle's frame

$$\vec{E}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

lab frame:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1-\frac{v^2}{c^2}}{\left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}\frac{\hat{R}}{R^2} =$$

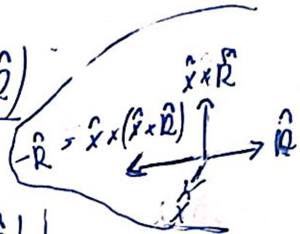
at the point of the second charge: $R = d$, $\hat{R} = \hat{y}$, $\theta = \frac{\pi}{2}$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1-\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\frac{\hat{y}}{d^2} = \frac{\gamma}{4\pi\varepsilon_0}\frac{q}{d^2}\hat{y}$$

$$\Rightarrow \vec{F}_e = q\vec{E} = \frac{\gamma}{4\pi\varepsilon_0}\frac{q^2}{d^2}\hat{y}$$

(2.) $\vec{B}_0 = 0$ (both particles at rest) → lab moving with $-\vec{v}$

$$\Rightarrow \vec{B} = -\frac{1}{c^2}\left(-\vec{v}\times\vec{E}\right) = -\frac{v}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{1-\frac{v^2}{c^2}}{\left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}\frac{(-\hat{x}\times\hat{R})}{R^2}$$



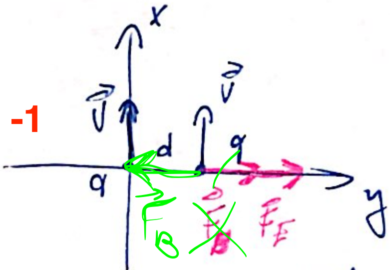
$$\vec{F}_B = q\left(\vec{v}\times\vec{B}\right) = q\left(+\frac{v^2}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{1-\frac{v^2}{c^2}}{\left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}\frac{\hat{x}\times(\hat{x}\times\hat{R})}{R^2}\right) =$$

$\theta = \frac{\pi}{2}$, $R = d$, $\hat{R} = \hat{y}$ →

$$= +\frac{v^2}{c^2}\frac{q^2}{4\pi\varepsilon_0}\frac{1-\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\frac{-\hat{y}}{d^2} = -\frac{v^2}{c^2}\frac{q^2}{4\pi\varepsilon_0}\frac{\gamma}{d^2}\hat{y} = -v^2\mu_0\varepsilon_0\frac{q^2}{4\pi\varepsilon_0}\frac{\gamma}{d^2}\hat{y}$$

$$= -\frac{\mu_0}{4\pi}\frac{v^2q^2\gamma}{d^2}\hat{y}$$

(3.) -1



(4.)
$$\vec{F} = q\left(\vec{E}+\vec{v}\times\vec{B}\right) = \vec{F}_E + \vec{F}_B = \left(\frac{\gamma}{4\pi\varepsilon_0}\frac{q^2}{d^2} - \frac{v^2}{c^2}\frac{q^2}{4\pi\varepsilon_0}\frac{\gamma}{d^2}\right)\hat{y} =$$

$$= \frac{\gamma}{4\pi\varepsilon_0}\frac{q^2}{d^2}\left(1-\frac{v^2}{c^2}\right)\hat{y} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{d^2}\frac{1-\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\hat{y} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{d^2}\sqrt{1-\frac{v^2}{c^2}}\,\hat{y} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{d^2\gamma}\hat{y}$$

$$\lim_{v\to c}\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{d^2}\sqrt{1-1} = 0 \Rightarrow \vec{F}_e \text{ and } \vec{F}_B \text{ balance out}$$

(5.) in particle system: $v = 0$, $\vec{B}_0 = 0$

$$\Rightarrow \vec{F}_0 = q\vec{E}_0 = q\frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\hat{y} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{d^2}\hat{y}$$

No

Question 5 13/14

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$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

(1.) $W = \frac{1}{2} L I^2$

$L = \frac{2W}{I^2}$

$L = \frac{2}{I^2} \frac{\mu_0 N^2 I^2 h}{4\pi} \ln\left(\frac{b}{a}\right)$

$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$

$W = \frac{1}{2\mu_0} \int B^2 \, d\tau \quad (E = 0)$

$|\vec{B}| = \begin{cases} \frac{\mu_0 N I}{2\pi s} & \text{inside the coil} \\ 0 & \text{outside} \end{cases}$

$= \frac{1}{2\mu_0} \iiint \frac{\mu_0^2 N^2 I^2}{4\pi^2 s^2} d\tau$

$= \frac{\mu_0 N^2 I^2}{8\pi^2} \int_0^h \int_0^{2\pi} \int_a^b \frac{1}{s^2} s \, ds \, d\phi \, dz$

$= \frac{\mu_0 N^2 I^2}{8\pi^2} \cdot 2\pi h \int_a^b \frac{1}{s} ds = \frac{\mu_0 N^2 I^2 h}{4\pi} \ln|s| \Big|_a^b =$

$= \frac{\mu_0 N^2 I^2 h}{4\pi} \ln\left(\frac{b}{a}\right)$

(2.) $I(t) = I_0 \cos(\omega t)$

R

$I_R(t) = ?$

$\hookrightarrow \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi} = \frac{\mu_0 I_0 \cos(\omega t)}{2\pi s} \hat{\phi}$

$I_R(t) = \frac{\mathcal{E}}{R} = \frac{-\frac{d\phi}{dt}}{R} = -\frac{1}{R} \frac{d}{dt} \int \vec{B} \cdot d\vec{a} =$

$= -\frac{1}{R} \frac{d}{dt} \int_0^h \int_a^b \frac{\mu_0 I_0 \cos(\omega t)}{2\pi s} ds \, dz =$

$= -\frac{1}{R} \frac{\mu_0 I_0}{2\pi} \frac{d \cos(\omega t)}{dt} h \int \frac{1}{s} ds = + \frac{\mu_0}{2\pi} \frac{h}{R} I_0 \sin(\omega t) \cdot \omega \cdot \ln\left(\frac{b}{a}\right)$

$I_R(t) = \frac{\mu_0}{2\pi} \frac{h \omega I_0}{R} \sin(\omega t) \ln\left(\frac{b}{a}\right)$

(3.) $\mathcal{E}_{back} = -L \frac{dI}{dt} = -\frac{\mu_0 N^2 h}{2\pi} \frac{\mu_0}{2\pi} \frac{h \omega I_0}{R} \ln\left(\frac{b}{a}\right)^2 \frac{d \sin(\omega t)}{dt} =$

$= -\frac{\mu_0^2 N^2 h^2 \omega^2 I_0}{4\pi^2 R} \cos(\omega t) \left[\ln\left(\frac{b}{a}\right)\right]^2$

(4.) the current in the wire (and magnetic field) mustn't be changing too rapidly and we can't be too far away from the wire. $r \ll \frac{c}{\omega}$ → the distance between the coil and the wire must be much smaller than $\frac{c}{\omega}$.

↳ we must be in the near zone

(5.) $\omega \ll \frac{c}{r}$, $r \sim 3$ cm

$\omega \ll \frac{3 \cdot 10^8}{3 \cdot 10^{-2}}$

$\omega \ll 10^{10}$ rad·s⁻¹

~~smaller~~ It works for angular frequencies much smaller than $10^{10}\ s^{-1}$, so up to ω of 10^8, $10^9\ s^{-1}$.